Model answers Exam Advanced Logic, 3-4-2018

1 a) 1) All propositional atoms p are wffs in $\mathcal{L}_\uparrow$

2) If A and B are wffs in $\mathcal{L}_\uparrow$, then so is $(A \uparrow B)$

3) Nothing is a wff in $\mathcal{L}_\uparrow$, except if it can be constructed in finitely many steps from 1, 2

b) i) Inductive definition of the translation operator $'$:

$p' = p$ for atoms p

$(\neg A') = A' \uparrow A'$

$(A \vee B)' = (A' \uparrow A') \uparrow (B' \uparrow B')$

$(A \wedge B)' = (A' \uparrow B') \uparrow (A' \uparrow B')$

$(A \rightarrow B)' = A' \uparrow (B' \uparrow B')$

$(A \leftrightarrow B)' = (A' \uparrow B') \uparrow ((A' \uparrow A') \uparrow (B' \uparrow B'))$

ii) To prove by induction that for every formula P in the language of propositional logic, P' is logically equivalent to P.

BASE For propositional atoms p, $p' = p$, so for all valuations v, $v(p') = v(p)$.

IH Let A and B be two arbitrary wffs of propositional logic and suppose that for all valuations v, $v(A') = v(A)$ and $v(B') = v(B)$.

Inductive step by cases! Take valuation v arbitrary

- $v((\neg A)') \underset{def^{\uparrow\prime}}{=} v(A' \uparrow A') \overset{def\ v}{=} v(\neg(A' \wedge A')) \overset{def\ v}{=} 1 - v(A') \overset{IH}{=} 1 - v(A) \underset{def\ v}{=} v(\neg A)$

- $v((A \vee B)') \underset{def\ '}{=} v((A' \uparrow A') \uparrow (B' \uparrow B')) \underset{def\ \uparrow}{=} v(\neg(\neg(A' \wedge A') \wedge \neg(B' \wedge B')))$

  $\underset{def\ v}{=} (1 - \min(v(\neg A'), v(\neg B'))) = (1 - \min(1 - v(A'), 1 - v(B')))$

  $= \cdot\cdot \max(v(A'), v(B')) \overset{IH}{=} \max(v(A), v(B)) = v(A \vee B)$

- $v((A \wedge B)') \underset{def\ '}{=} v((A' \uparrow B') \uparrow (A' \uparrow B')) \underset{def\ \uparrow}{=} v(\neg(\neg(A' \wedge B') \wedge \neg(A' \wedge B')))$

  $\underset{def\ v}{=} v(\neg\neg(A' \wedge B')) \underset{def\ v}{=} v(A' \wedge B') = \min(v(A'), v(B')) \overset{IH}{=}$

  $\min(v(A), v(B)) \underset{def\ v}{=} v(A \wedge B)$

$1^{b),\,ii)}$ continued

- $v((A\to B)') \underset{def\,'}{=} v(A'\uparrow(B'\uparrow B')) \underset{def\,\uparrow}{=} v(\neg(A'\land\neg(B'\land B'))) =$

$$= v(\neg(A'\land\neg B')) = 1 - v(A'\land\neg B') =$$
$$1-\min(v(A'),\ 1-v(B')) \overset{IH}{=} 1-\min(v(A), 1-v(B)) =$$
$$\max(v(\neg A), v(B)) = v(A\to B)$$

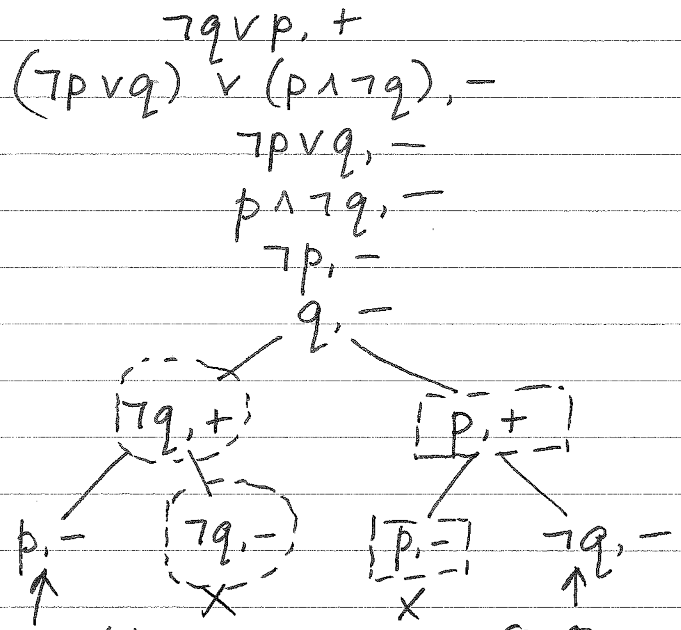
- $v((A\leftrightarrow B')) \underset{def\,'}{=} v((A'\uparrow B')\uparrow((A'\uparrow A')\uparrow(B'\uparrow B'))) \underset{def\,\uparrow}{=}$

$$v(\neg[(A'\uparrow B')\land\neg(\neg A'\land\neg B')]) =$$
$$v(\neg[\neg(A'\land B')\land\neg(\neg A'\land\neg B')]) =$$
$$v((A'\land B')\lor(\neg A'\land\neg B')) =$$

$$\max(\min(v(A'), v(B')), \min(1-v(A'), 1-v(B')) \overset{IH}{=}$$
$$\max(\min(v(A), v(B)), \min(1-v(A), 1-v(B)) =$$
$$v(A\leftrightarrow B)$$

Conclusion: Therefore, for all propositional wffs A and all valuations v, $v(A') = v(A)$, i.e., A' and A are logically equivalent

3. To test whether $\neg q \vee p \vdash_{K_3} (\neg p \vee q) \vee (p \wedge \neg q)$ is valid, we make a tableau:



① open, complete

② open, complete

The tableau has open complete branches, so the inference is not valid.

Counterexample ①: $q\rho 0$, nothing obtains about p

Counterexample ②: $p\rho 1$, nothing obtains about q

2. Truth table to test whether $\vDash_{RM_3} (p \supset q) \vee (q \supset p)$:

p	q	(p ⊃ q)	∨	(q ⊃ p)
1	1	1	1	1
1	i	0	1	1
1	0	0	1	1
i	1	1	1	0
i	i	i	i	i
i	0	0	1	1
0	1	1	1	0
0	i	1	1	0
0	0	1	1	1

Conclusion: Under all valuations v, $v((p \supset q) \vee (q \supset p)) \in \{i, 1\}$, which is the set of designated values of RM_3, so the inference holds.

5. To test whether $\Diamond\Box\Diamond p \vdash_K \Diamond\Box p$ is valid, we make a tableau:

$\Diamond\Box\Diamond p, 0$

$\neg\Diamond\Box p, 0$

$|$

$0r1$

$\Box\Diamond p, 1$

$\Box\neg\Box p, 0$

$\neg\Box p, 1$

$\Diamond\neg p, 1$

$1r2$

$\neg p, 2$

$\Diamond p, 2$

$2r3$

$p, 3$

$\uparrow$

open, complete.

There is an open, complete branch, so the inference is not valid. We can read off a countermodel from the branch: $I = \langle W, R, v\rangle$ with

$W = \{w_0, w_1, w_2, w_3\}$

$R = \{\langle w_0, w_1\rangle, \langle w_1, w_2\rangle, \langle w_2, w_3\rangle\}$

$v_{w_2}(p) = 0, \quad v_{w_3}(p) = 1$

valuation in w_0, w_1 can be freely chosen.

4. The inference $(p\wedge q)\to r \models_{0.6} (p\to r)\to(q\to r)$ is not valid.
Take, for example, the interpretation v (with $v(p) = 0.3$, $v(q) = 1$, and $v(r) = 0.5$.

Then $v(p\wedge q) = 0.3$, so $v(p\wedge q) \leq v(r)$, so $v((p\wedge q)\to r) = 1$.

Also,
- $v(p) \leq v(r)$, so $v(p\to r) = 1$.
- $v(q) > v(r)$, so $v(q\to r) = 1 - (v(q) - v(r)) = 1 - 0.5 = 0.5$

Therefore, $v((p\to r)\to(q\to r)) = 1 - (1 - 0.5) = 0.5$

Conclusion: $v((p\wedge q)\to r) = 1 \geq 0.6$ while $v((p\to r)\to(q\to r)) = 0.5 < 0.6$, so the inference is not valid.

6. To test whether $\langle F\rangle[P]q \vdash^{t}_{K_{\delta\tau}} \langle F\rangle q \wedge [P][P]q$ is valid, we make a tableau:

$\langle F\rangle[P]q, 0$

$\neg(\langle F\rangle q \wedge [P][P]q), 0$

$0r1$

$[P]q, 1$

$0r2$ (δ)

$2r1$

$q, 2$

$\neg\langle F\rangle q, 0$

$[F]\neg q, 0$

$\neg q, 2$

×

$\neg[P][P]q, 0$

$\langle P\rangle\neg[P]q, 0$

$3r0$

$\neg[P]q, 3$

$\langle P\rangle\neg q, 3$

$4r3$

$\neg q, 4$

$4r0$ (τ)

$4r1$ (τ)

$q, 4$

×

The tableau is closed, so the inference is valid.

7. 1. $\neg(\Diamond\Diamond p \supset \Diamond p), 0$
2. $\Diamond\Diamond p \ , 0$
3 $\neg\Diamond p \ , 0$
4 $\Box\neg p \ , 0$
5 $0 r 1$
6 $\Diamond p, 1$
7 $1 r 2$
8 $p, 2$
9 $\neg p, 1$
10 $1 r 0$
11 $2 r 1$

Branch b is (part of) of a K_σ - tableau.

(a) b is complete, because all rules that can be applied, have been applied, as follows:

line: 1. The only rule applicable to l. 1 is the $\neg\supset$ rule. The result is in l. 2, 3

2 The only rule applicable to l. 2 is the $\Diamond$ rule. The result is in l. 5, 6

3. The only rule applicable to l. 3 is the $\neg\Diamond$ rule. The result is in l. 4.

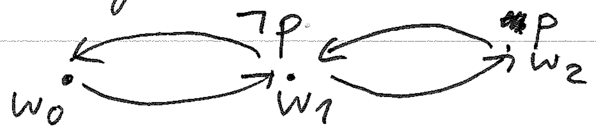
6. The only rule applicable to l. 6 is the $\Diamond$ rule. The result is in l. 7, 8

4. The rule applicable to l. 4 is the $\Box$ rule. There is only one i with $0 r i$ on the branch, namely, $i = 1$. The result is l. 9

5 The only rule applicable to l. 5 is σ. Result is l. 10

7 The only rule applicable to l. 7 is σ. Result is l. 11

(b) $I = \langle W, R, v\rangle$ with $W = \{w_0, w_1, w_2\}$, $R = \{\langle w_0, w_1\rangle, \langle w_1, w_2\rangle, \langle w_1, w_0\rangle, \langle w_2, w_1\rangle\}$. $v_{w_2}(p) = 1$, $v_{w_1}(p) = 0$ (& freely chosen in w_0).
Define function f with $f(0) = w_0$, $f(1) = w_1$ and $f(2) = w_2$.
Now it is clear that for all i, j : irj on $b \Rightarrow f(i) R f(j)$ in I.
Also, we check line by line that:
for every node D, i on b, D is true at world $f(i)$ in I.



7 b) Continued

l. 9: $\neg p, 1$ on b and indeed $v_{w_1}(p) = 0$.

l. 8: $p, 2$ on b and indeed $v_{w_2}(p) = 1$

l. 6: $\Diamond p, 1$ on b and indeed $v_{w_1}(\Diamond p) = 1$ because $w_1 R w_2$ and $v_{w_2}(p) = 1$

l. 4: $\Box \neg p, 0$ on b and indeed $v_{w_0}(\Box \neg p) = 1$ because only world w_1 is accessible by R from w_0, and $v_{w_1}(\neg p) = 1$

l. 3: $\neg \Diamond p, 0$ on b and indeed $v_{w_0}(\neg \Diamond p) = 1$ because $v_{w_0}(\Box \neg p) = 1$ (see l.4)

l. 2: $\Diamond\Diamond p, 0$ on b and indeed $v_{w_0}(\Diamond\Diamond p) = 1$ because $w_0 R w_1$ and $v_{w_1}(\Diamond p) = 1$ (see l. 6)

l. 1: $\neg(\Diamond\Diamond p \supset \Diamond p), 0$ on b and indeed $v_{w_0}(\neg(\Diamond\Diamond p \supset \Diamond p)) = 1$, because $v_{w_0}(\Diamond\Diamond p) = 1$ (l.2) and $v_{w_0}(\neg \Diamond p) = 1$ (l.3)

8 To test whether $\Box \forall x Px \wedge \Diamond \exists x Qx \vdash_{VK} \Diamond \exists x (Px \wedge Qx)$ is valid, we make a tableau:

$$\begin{array}{c}
\Box \forall x Px \wedge \Diamond \exists x Qx, 0 \\
\neg \Diamond \exists x (Px \wedge Qx), 0 \\
\downarrow \\
\Box \forall x Px, 0 \\
\Diamond \exists x Qx, 0 \\
\Box \neg \exists x (Px \wedge Qx), 0 \\
\downarrow \\
0 r 1 \\
\exists x Qx, 1 \\
\mathcal{E} a, \underline{1} \\
Qa, 1 \\
\forall x Px, 1 \\
\begin{array}{cc}
/ & \backslash \\
\begin{array}{c} \neg \mathcal{E} a, 1 \\ \times \end{array} &
\begin{array}{c}
Pa, 1 \\
| \\
\neg \exists x (Px \wedge Qx), 1 \\
\forall x \neg (Px \wedge Qx), 1 \\
\begin{array}{cc}
/ & \backslash \\
\begin{array}{c} \neg \mathcal{E} a, 1 \\ \times \end{array} &
\begin{array}{c}
\neg (Pa \wedge Qa), 1 \\
\begin{array}{cc}
/ & \backslash \\
\begin{array}{c} \neg Pa, 1 \\ \times \end{array} & \begin{array}{c} \neg Qa, 1 \\ \times \end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}$$

All branches close so the tableau is closed and the inference is valid.

9. a) i) (δ_2) is a process, because δ_2 can be applied to $In() = Th(W) = Th(\{A(d), L(d)\})$: namely, $L(d_1) \wedge A(d_1) \in Th(W)$, while $\neg T(d) \notin Th(W)$.
The process (δ_2) is not closed, because δ_1 is applicable to $In(\delta_2) = Th(\{A(d), L(d), T(d)\})$: namely, $L(d_1) \in In(\delta_2)$, while $\neg M(d) \notin In(\delta_2)$.
The process δ_2 is successful, because $In(\delta_2) \cap Out(\delta_2) = Th(\{A(d), L(d), T(d)\} \cap \{\neg T(d)\} = \emptyset$.

ii) (δ_2, δ_1) is a process, because δ_2 is applicable to $In()$ and δ_1 is applicable to $In(\delta_2)$ [see (i)].
(δ_2, δ_1) is closed because δ_3 is not applicable to $In(\delta_2, \delta_1) = Th(\{A(d), L(d), T(d), M(d)\}$. This is because $\neg(\neg A(d) \vee \neg M(d)) (\Leftrightarrow A(d) \wedge M(d)) \in In(\delta_2, \delta_1)$.
(δ_2, δ_1) is successful because $In(\delta_2, \delta_1) \cap Out(\delta_2, \delta_1) = Th(\{A(d), L(d), T(d), M(d)\}) \cap \{\neg M(d), \neg T(d)\} = \emptyset$.

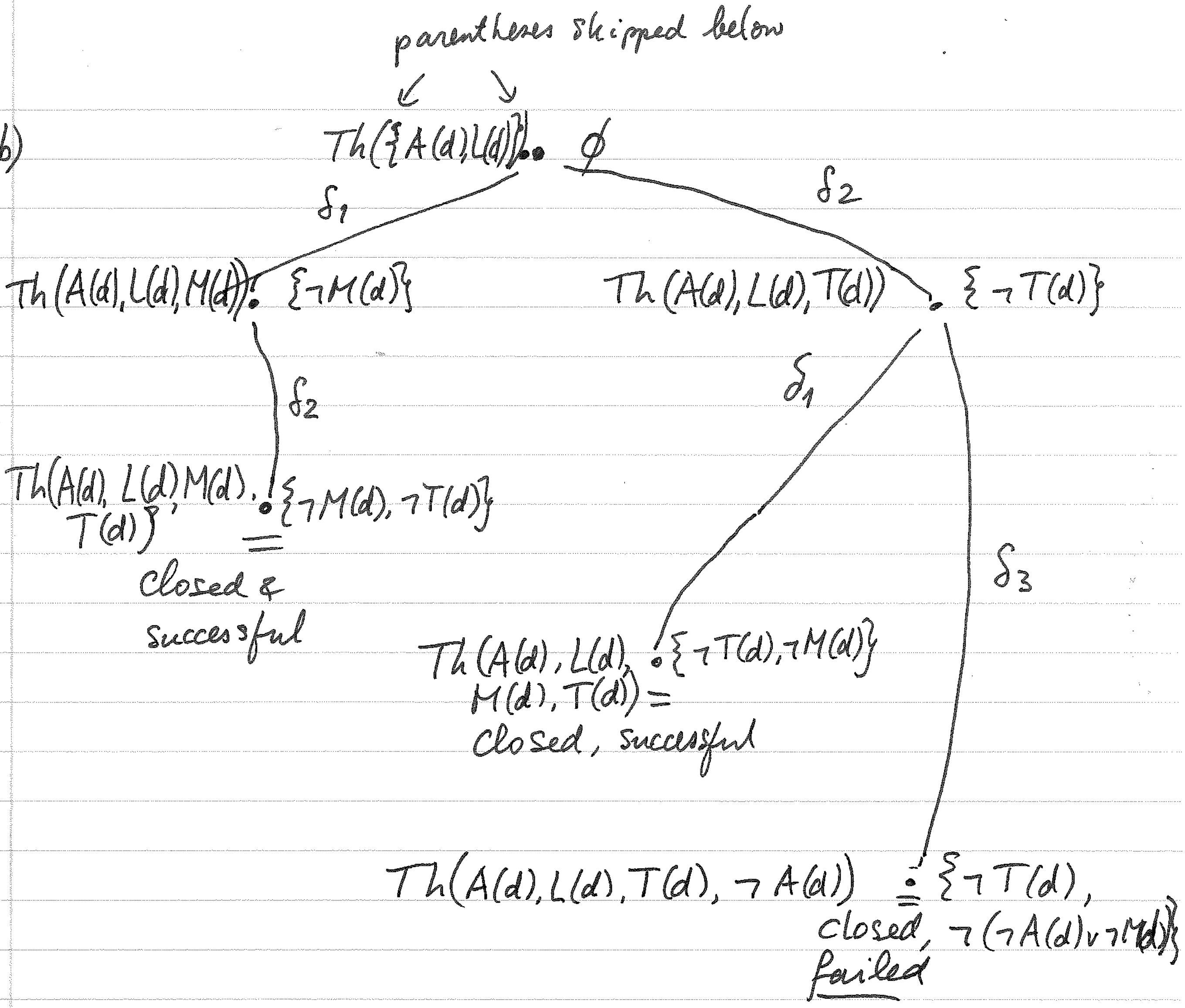
iii) To check whether (δ_2, δ_3) is a process, we need to check two things:

- δ_2 is applicable to $In()$ (see i)
- δ_3 is applicable to $In(\delta_2) = Th(\{A(d), L(d), T(d)\})$ because $M(d) \vee T(d) \in In(\delta_2)$, while $\neg(\neg A(d) \vee \neg M(d)) (\Leftrightarrow A(d) \wedge M(d)) \notin In(\delta_2)$.

So indeed, (δ_2, δ_3) is a process.
However, $In(\delta_2, \delta_3) = Th(\{A(d), L(d), T(d), \neg A(d)\})$.
This is an inconsistent set, therefore it is:

- closed, because $\neg M(d) \in In(\delta_2, \delta_3)$ (everything is in an inconsistent theory)
- unsuccessful, because $In(\delta_2, \delta_3) \cap Out(\delta_2, \delta_3) = \{\neg M(d), \neg T(d)\}$ (again because an inconsistent theory contains all sentences), so it is not $\emptyset$.

9b)



c) There is only one extension, namely $In(\delta_2, \delta_1) = In(\delta_1, \delta_2) = Th(\{A(d), L(d), M(d), T(d)\})$, correspondng to the two closed successful processes.

10.) No, the statement does <u>not</u> hold. Take $A: p \wedge \neg p$; $B: q \vee \neg q$. Then $p \wedge \neg p \vdash_{K_3} q \vee \neg q$ and $p \wedge \neg p \vdash_{LP} q \vee \neg q$, but <u>not</u> $p \wedge \neg p \vdash_{FDE} q \vee \neg q$. Let's make a tableau:

$$p \wedge \neg p, +$$
$$q \vee \neg q, -$$
$$p, +$$
$$\neg p, +$$
$$q, -$$
$$\neg q, -$$

- The branch is closed for K_3, because it contains both $p, +$ and $\neg p, +$. So $p \wedge \neg p \vdash_{K_3} q \vee \neg q$.
- The branch is closed for LP, because it contains $q, -$ and $\neg q, -$, so $p \wedge \neg p \vdash_{LP} q \vee \neg q$.
- The branch is open & complete for FDE. Counter-model: $p\rho 1$, $p\rho 0$, nothing obtains about q